SEMI-ANNUAL STATUS REPORT

on

NASA GRANT NsG-576

(February 1, 1965 to July 31, 1965)

Submitted to

Office of Grants and Research Contracts
National Aeronautics and Space Administration
Washington, D. C.

by

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September, 1965

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INTRODUCTION

The main objective of this research project is to develop the analytical and numerical methods for the elastic and viscoelastic analysis of two-dimensional problems with star-shaped boundaries. The analytical method is based on the theory of complex variables; therefore, it requires the mapping functions for star-shaped regions. The research work on the derivation of mapping functions, which has been conducted since February of 1964, resulted into the following publications:

Rim, K., and Stafford, R. O., "Derivation of Mapping Functions for Star-Shaped Regions," NASA CR-192, NASA, March, 1965.

Rim, K., and Camillo, L., "On Mapping Functions for Torsional Analysis of Splined Shafts," the 9th Midwestern Mechanics Conference, August, 1965.

Besides these, the Supplement to NASA CR-192, which is an extension of NASA CR-192 for interior-to-interior mapping, is almost completed and will soon be submitted for publication. Another report on an investigation and development of various methods of approximate conformal transformation is expected to be ready in about two to three months.

The starting date of the research on the numerical method based on potential theory coincided with the beginning of the period to be covered by the present status report. Therefore, as far as the development of the numerical method is concerned, the summary of its entire progress will be presented as a main part of this status report without any introduction.

During the period covered by this status report, research efforts were concentrated on

a) Completion of the simple method for interior-to-interior mapping.
b) Investigation and development of various methods of approximate conformal transformation, and

c) Development of a direct numerical technique, based on the integral equation method in potential theory, for the solution of general bi-harmonic boundary value problems with simply connected domains.

These three areas of effort will be explained in more detail in the following text.

INTERIOR-TO-INTERIOR MAPPING

A simple method for mapping the interior of a unit circle onto the interior of a given region has been developed. Since it is similar to NASA CR-192 and may be regarded as an extension of it, it is being prepared for publication as a Supplement to NASA CR-192. No further details will be elaborated here, since it will soon be submitted to NASA for publication.

To illustrate the application of this method, a paper entitled "On Mapping Functions for Torsional Analysis of Splined Shafts" was presented at the 9th Midwestern Mechanics Conference. Its preprints were already submitted to NASA as a part of this status report.

DEVELOPMENT OF VARIOUS APPROXIMATE MAPPING TECHNIQUES

The successful development of an analytical method hinges on the construction of appropriate mapping functions. Therefore, a major portion of the research effort has been concentrated on the development of various mapping techniques. A number of promising methods of approximate conformal transformation were explored in depth, searching for the simplest and most effective method of mapping.

In the previous report (ref. 6) and its supplement, the authors developed an effective method of constructing the appropriate mapping functions: this was based on the well-known Schwarz-Christoffel transformation. Also available now is another technique developed by others (ref. 1 and 2) which is based on the integral equations set forth in the book by Kantorovich and Krylov (ref. 3). Besides these, three additional methods have been developed in the course of our investigation for simpler mapping techniques. Starting from some basic principles stated in their general form in reference 3, practical techniques have been developed and are computer programmed for ready application. Each method will be briefly described in the following pages.

1. Mapping Functions Derived from the Property of Minimum Area.

Let $\zeta = f(z)$ be a function which transforms the interior of a simply connected region onto the interior of a circle. The desired function $f(z)$ which accomplishes this transformation gives a minimum value for the area in the ζ-plane (ref. 3); i.e., the integral

$$I = \iint_D f'(z)\, \overline{f'(z)}\, dxdy, \tag{1}$$

subject to the condition that $f'(0) = 1$, takes on a minimum value. Thus finding $f'(z)$ can be reduced to a problem of variational calculus (ref. 5).

Now consider the specific case of a star-shaped region with $2m$ axes of symmetry.

Figure 1. Mapping of a star-shaped region onto a circle.

Assume $f'(z)$ to be a truncated lacunary series,

$$f'(z) = \sum_{j=0}^{n} C_j \, z^{jm}, \quad C_0 = 1, \quad z = R(\phi)e^{i\phi}, \quad i = \sqrt{-1}. \qquad (2)$$

Then the function to be minimized becomes

$$\frac{I}{2m} = \sum_{j=0}^{n} \sum_{k=0}^{n} \frac{C_j \quad C_k}{(j+k)\;m+2} \int_0^{\pi/m} R^{(j+k)m+2} \, \mathrm{Cos}(j-k)m\phi d\phi, \qquad (3)$$

where simplifications due to symmetry have been introduced. Upon minimization, one arrives at n equations for the n unknowns C_j;

$$0 = \alpha_{0,k} + \sum_{j=1}^{n} C_j \, \alpha_{j,k}, \quad k = 1, 2, \ldots, n,$$

$$\alpha_{j,k} = \frac{1}{(j+k)\;m+2} \int_0^{\pi/m} R^{(j+k)m+2} \, \mathrm{Cos}(j-k)m\phi d\phi. \qquad (4)$$

Hence, the mapping function is given by the integration of equation (2):

$$\zeta = \int f'(z)dz = \sum_{k=0}^{n} \frac{C_k}{1+km} z^{1+km} = \sum_{k=0}^{n} A_k \, z^{1+km} \qquad (5)$$

A similar minimum property holds for exterior regions. The result is that one may replace m by $-m$ in equations (2) through (5).

2. Mapping Functions Derived from the Property of Minimum Contour Length.

Let $\zeta = f(z)$ be a function which transforms a simply connected region onto a circle. The function which accomplishes this transformation gives a minimum value to the length ℓ, the image of the contour L; see figure (1) and reference (3). This contour length is given by the integral

$$I = \int_{\ell} |\zeta'|ds = \int_{L} |f'(z)|dS;$$

where $f(0) = 0$ and $f'(0) = 1$ for interior regions, and $f'(\infty)$ is finite for exterior regions. Thus finding $f(z)$ can again be reduced to a variational problem.

It is convenient to find approximations to $\sqrt{f'(z)}$ rather than to $f'(z)$. Thus define

$$\phi(z) = \sqrt{f'(z)} = \sum_{j=0}^{n} B_j\, Z^{jm},\ B_0 = I,\quad z = R(\phi)e^{i\phi}, \tag{6}$$

where again the specific case of a star-shaped region with 2m axes of symmetry is considered. Note that m is positive for interior-to-interior and negative for exterior-to-exterior mapping, respectively.

Hence the function to be minimized becomes

$$I = \int_{L} |\phi(z)|^2 dS = \int_{L} \phi(z)\overline{\phi(z)}\, dS$$

$$= 2m \sum_{j=0}^{n} \sum_{k=0}^{n} B_j B_k \int_0^{\pi/m} R^{(j+k)m} \sqrt{\frac{dR}{d\phi}^2 + R^2}\ \mathrm{Cos}(j-k)m\phi d\phi,$$

where simplifications due to symmetry have been introduced. The minimization of $I/2m$ yields n equations for the n unknowns B_j;

$$0 = \beta_{0,k} + \sum_{j=1}^{n} B_j \cdot \beta_{j,k}, \qquad k = 1, 2, \ldots, n,$$

$$\beta_{j,k} = \int_0^{\pi/m} R^{(j+k)m} \sqrt{\frac{dR}{d\phi}^2 + R^2}\ \mathrm{Cos}(j-k)m\phi d\phi. \tag{7}$$

The desired approximate mapping function is given by

$$f(z) = \int \phi^2(z)dz = \sum_{k=0}^{2n} A_k Z^{1+km}, \quad A_k = \frac{\sum_{j=0}^{k} B_j \cdot B_{k-j}}{1 + km}. \tag{8}$$

3. Mapping Functions Derived from Successive Approximations.

Let $z = f(\zeta)$ be a polynomial mapping function which transforms a unit circle onto a simply connected star-shaped region. A peculiar characteristic of polynomial mapping functions is that $|\phi - \theta| << \theta$ or ϕ. Hence $\phi - \theta$ may be approximated by a few terms of a suitably chosen series. Since $\phi - \theta$ must be zero at each axis of symmetry, one may take the following form

$$\phi - \theta = \sum_{j=1}^{\ell} A_j \text{ Sin } jm\theta. \tag{9}$$

Upon dividing the mapping function by ζ, one obtains

$$z/\zeta = R(\phi)e^{i(\phi-\theta)} = \sum_{k=0}^{n} A_k \zeta^{km}, \; \zeta = e^{i\theta}. \tag{10}$$

First, suppose that $\phi - \theta$ is a known function. Then the constants Ak are simply the Fourier coefficients of equation (10);

$$A_k = \frac{m}{\pi} \int_0^{\pi/m} R(\phi) \{\text{Cos}(\phi-\theta) \text{ Cos } km\theta - \text{Sin}(\phi-\theta) \text{ Sin } km\theta\} d\theta. \tag{11}$$

where $(\phi - \theta)$ is given by equation (9) and $R(\phi)$ is assumed to be an even function of θ.

Assuming the mapping function is known, one can determine the coefficients A_j in equation (9). Equating the imaginary parts of the logrithum of equation (10), one obtains

$$(\phi - \theta) = \tan^{-1}\left\{\frac{\sum_{k=1}^{n} A_k \text{ Sin } km\theta}{\sum_{k=0}^{n} A_k \text{ Cos } km\theta}\right\} = F(\theta). \tag{12}$$

Hence A_j are the Fourrier coefficients of $(\phi - \theta)$;

$$A_k = \frac{2m}{\pi}\int_0^{\pi/m} F(\theta) \text{ Sin } km\theta d\theta. \tag{13}$$

Equations (11) and (13) may therefore be used as predictor and corrector formulas in an iteration procedure for determining the mapping function.

The three approximate methods introduced here require a digital computer for their application to any but the simplest problems. Equations (4), (7), (11) and (12) require the integration of a slowly varying function multiplied by a rapidly varying function. This can be accomplished quite easily on a computer by employing a variation of Filon's quadratures (ref. 4) and defining the boundary in terms of straight lines and circular arcs. Because of symmetry (i.e., $\beta_{ij} = \beta_{ji}$) the number of required computations is reduced by nearly one-half.

The two methods based on certain minimum properties present some difficulties in their application. First, they require the radius $R(\phi)$ to be raised to a very high (or low) power in the course of computation, resulting in ill-conditioned matrices. Therefore, the application of these methods is restricted to the derivation of mapping functions in the form of low-order polynomials. It was also found that the dimensions must be carefully scaled to maintain the computational accuracy. Another problem was that of reverting the derived function $\zeta = f(z)$ to $z = f(\zeta)$. But it was quickly overcome by developing a simple procedure for reverting a general lacunary power series.

In view of its generality and simplicity, the method of successive approximation seems to be the most advantageous of the approximate methods. It is not restricted to a polynomial with a small number of terms and does not require the inversion of a power series. Required computations are simple and may be performed on any small-size digital computer or even a desk calculator.

DEVELOPMENT OF AN EFFECTIVE NUMERICAL METHOD IN PLANE ELASTICITY

By means of the integral equation method in potential theory, a general numerical method of solving the following elasticity problems has been formulated for simply-connected domains:

a) Interior problems,

b) Exterior problems.

For each case, one typical problem with known analytic solution has been solved for the purpose of illustration. The numerical values of the stress fields obtained by this method compare very favorably with the exact analytic solutions.

For a typical interior problem, a circular disk subjected to a pair of diametrally opposite concentrated loads was analyzed. Due to symmetry, it was necessary to consider only one quadrant of the circular disk. The stress computations were carried out by using an IBM 7044 computer in a single precision arithmetic, with boundary subdivisions of 4, 8, 16 and 32 intervals. Although this problem is considered to be an extreme loading condition, the numerical values of the stress field generated from the case of 32 boundary subdivisions agrees well with the exact solution.

A plate with an elliptic aperture subjected to a uniform normal pressure was chosen as a typical exterior problem. By varying the semi-axis ratio of the elliptic hole, it was possible to investigate the effect of the boundary geometry on the accuracy of the numerical solutions. From the indications given so far, it appears that the accuracy is not significantly affected by the complexity of the boundary configuration, provided it is smooth. Stress computations were carried out also for 32 boundary subdivisions by using the IBM 7044 computer in a single precision arithmetic. The numerical results compare favorably with the exact analytic solution.

Since we will soon prepare a report on this integral equation method, no further details (numerical results, tables, graphs, etc.) will be given in the present status report. The report will cover our research work on the formulation and analysis of simply-connected domain problems. The results of our past six-month research indicate that this new numerical technique should provide a powerful method of solving some of the complicated boundary-value problems which are not amenable to any other treatment. Due to its generality, a successful extension to multiply connected domains will exert a far-reaching influence on every area of mechanics and mathematical physics.

RESEARCH FOR THE SUCCEEDING HALF YEAR

In the area of developing an analytical technique based on the complex variable method, our research effort will be directed to the following projects:

a) Preparation of a comprehensive report for publication on conformal mapping,

b) Continuation of the research work on the formulation of linear elastic and viscoelastic analysis of star-shaped regions.

Besides these, a formulation of the non-linear elastic analysis is planned to be initiated at the later part of the period.

The development of a general numerical method for solving bi-harmonic boundary-value problems will be carried out in the following order:

a) Applying the formulation for simply-connected domain problems; exterior problems with star-shaped boundaries will be analyzed,

b) A comprehensive report on the analysis of simply-connected domain problems will be prepared for publication,

c) Formulation for doubly-connected domain problems will be initiated, with particular emphasis on the development of a general method of analyzing the stresses in the solid propellent rocket motors.

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